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ON THE ALGORITHM OF THE SOLUTION OF THE SIGNORINI PROBLEM*

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An algorithm is proposed for solving the Signorini problem /1/ in the formulation of a unilateral variational problem for the boundary functional in the zone of possible contact /2/. The algorithm is based on a dual formulation of Lagrange maximin problems for whose solution a decomposition approach is used in the following sense: a Ritz process in the basis functions that satisfy the linear constraint of the problem, the differential equation in the domain, is used in solving the minimum problem (with fixed Lagrange multipliers); the maximum problem is solved by the method of descent (a generalization of the Frank-Wolf method) under convexity constraints on the Lagrange multipliers. The algorithm constructed can be conisidered as a modification of the well-known algorithm to find the Udzawa-Arrow-Hurwitz saddle points /3, 4/. The convergence of the algorithm is investigated. A numerical analysis of the algorithm is performed in the example of a classical contact problem about the insertion of a stamp in an elastic half-plane under approximation of the contact boundary by isoparametric boundary elements. The comparative efficiency of the algorithm is associated with the reduction in the dimensionality of the boundary value problem being solved and the possibility of utilizing the calculation apparatus of the method of boundary elements to realize the solution.

1. Solution of the generalized Signorini problem in the domain $G \subset E_3$ with a sufficiently smooth boundary S reduces /2/ to solving a variational problem for the boundary functional

$$F(\varphi) = \frac{1}{2}\int_{S_1} \mathbf{t}^{(\nu)}(\varphi)\,\varphi\,ds + \int_{S_1} \mathbf{t}^{(\nu)}(\mathbf{u}^*)\,\varphi\,ds \tag{1.1}$$

on the boundary of possible contact $S_1 \subset S$ with unit internal normal vector (ν). The functional $F(\varphi)$ is determined in a convex closed set /2/

$$\mathbf{V}^*(S_1) = \{\varphi \in \mathbf{W}_2^{*1/2}(S_1) \mid \varphi^{(\nu)}|_{S_1} \geqslant 0\} \tag{1.2}$$

where $\mathbf{W}_2^{*1/2}(S_1) \subset \mathbf{W}_2^{1/2}(S_1)$ is a subspace of traces of the displacement vector $\varphi(x)$, $x \in \bar{G}$ on S_1 that satisfy the linear constraints of the variational problem for $F(\varphi)$ in the form of the equalities

$$\mathbf{A}\varphi = 0 \text{ in } G, \quad \mathbf{t}^{(\nu)}(\varphi)|_{S_2} = 0, \quad S_2 = S \mid S_1 \tag{1.3}$$

and the conditions

$$\int_G \varphi\,dG = \int_G \operatorname{rot}\varphi\,dG = 0$$

(the smoothness of the boundary S is here and henceforth assumed to be such that the theorem on traces holds). By virtue of the Betti formula /5/, we have for such vector-functions

$$\int_{S_1} \mathbf{t}^{(\nu)}(\varphi)\,\varphi\,ds = 2\int_G W(\varphi)\,dG > 0, \quad \mathbf{V}\varphi \neq 0$$

$$\int_{S_1} \mathbf{t}^{(\nu)}(\varphi)\,\varphi\,ds = 0 \Leftrightarrow \varphi = 0$$

($W(\varphi)$ is the quadratic form of linear elasticity theory /5/) so that the boundary norm in $\mathbf{W}_2^{*1/2}(S_1)$ is taken equal to /2/

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$$|\varphi| = \left\{\int_S \mathbf{t}^{(\nu)}(\varphi)\varphi\, ds\right\}^{1/2} \tag{1.4}$$

The unique solvability of the problem of minimizing the functional (1.1) in the set $\mathbf{V}^*(S_1)$ is proved in /2/ and it is established that its solution $\varphi_0 \in \mathbf{V}^*(S_1)$ is a solution of the following unilateral boundary value problem for the displacement vector φ_0:

$$\begin{gathered} \mathbf{A}\varphi_0 = 0 \text{ in } G, \quad \varphi_0^{(\nu)}|_{S_1} \geqslant 0, \quad [\mathbf{t}^{(\nu)}(\varphi_0) + \mathbf{t}^{(\nu)}(\mathbf{u}^*)]_{S_1} \geqslant 0 \\ \varphi_0^{(\nu)}[\mathbf{t}^{(\nu)}(\varphi_0) + \mathbf{t}^{(\nu)}(\mathbf{u}^*)]_{S_1} = 0, \quad \mathbf{t}^{(\nu)}(\varphi_0)|_{S_2} = 0 \end{gathered} \tag{1.5}$$

Here and above $\mathbf{t}^{(\nu)}(\mathbf{u}^*)$ is a given surface stress vector in the zone of possible contact S_1. The displacement vector $\mathbf{u}^*(x)$, $x \in \bar{G}$, which is considered to be known, is /2/ a solution of the auxiliary mixed problem of elasticity theory with zero boundary condition for $\mathbf{u}^*$ in the zone of possible contact. The mechanical interpretation of problem (1.5) corresponds to the problem of the equilibrium of an elastic body $\bar{G}$ resting on a certain stiff surface without friction at points of the boundary of S_1 and subjected to surface stresses $\mathbf{t}^{(\nu)}(\mathbf{u}^*)$ in the zone of possible contact when there are no mass forces.

A dual variational problem to the problem of minimizing the functional $F(\varphi)$ on $\mathbf{V}^*(S_1)$ is formulated in /2/ by using the Young- Fenchel-Moro transformation /4/ in terms of the surface stresses in the zone of possible contact. The difficulty of a practical realization of the solution of this problem is due to the difficulty in constructing the function of the dual problem in explicit form. Consequently, the method of Lagrange multipliers is used to formulate the dual problem below.

2. Furthermore, to simplify the discussion we will consider the Signorini problem for a second-order scalar elliptic operator with symmetric bilinear $B(u, v)$ and positive-definite quadratic form $B(v)$ (see /1/, p.115), in which case the results are extended in a natural manner to the formulation of the Signorini problem for the linear elasticity theory operator of Sect.1.

The convexity constraint is given by a closed convex set of scalar functions (similar to (1.2)) defined on the whole domain boundary

$$V^*(S) = \{v \in W_2^{*1/2}(S) \mid v|_S \geqslant 0\}$$

where $W_2^{*1/2}(S) \subset W_2^{1/2}(S)$ is a subspace of traces of the scalar functions v in S that satisfy the equation $Av = 0$ in G; the norm in the subspace $W_2^{*1/2}(S)$ is defined /6/ by the expression (similar to (1.4))

$$\|v\|_{1/2,\, S} = \left\{\int \partial_{\nu_A} v v\, ds\right\}^{1/2} \tag{2.1}$$

(here and henceforth, unless otherwise stated, the integration is over the boundary S), and $\partial_{\nu_A} = \partial/\partial\nu_A$ is differentiation with respect to the direction of the conormal ν_A (the subscript A is henceforth omitted).

The Signorini problem in the formulation presented in /1/ can be reduced, as in /2/, to a minimization problem for the functional

$$F_0(\varphi) = {}^1/_2 \int \partial_\nu \varphi\varphi\, ds + \int \partial_\nu u^* \varphi\, ds \tag{2.2}$$

in the set $V^*(S)$.

The solution $\varphi_0 \in V^*(S)$ of the variational problem for the functional $F_0(\varphi)$ is the solution of the following unilateral boundary value problem (similar to (1.5)):

$$\begin{gathered} A\varphi_0 = 0 \text{ in } G, \quad \varphi_0|_S \geqslant 0, \quad [\partial_\nu\varphi_0 + \partial_\nu u^*]_S \geqslant 0 \\ \varphi_0[\partial_\nu\varphi_0 + \partial_\nu u^*]_S = 0 \end{gathered} \tag{2.3}$$

Since $F_0(\varphi)$ is a strictly convex functional and $V^*(S)$ is a convex closed set in $W_2^{*1/2}(S)$, the problem in finding

$$\inf_{\varphi \in V^*(S)} F_0(\varphi) \tag{2.4}$$

is solvable uniquely /7/.

We will now formulate the dual problem and we present (with or without proof) assertions corresponding to theorems for the equivalence and existence of a saddle point /7/.

Let

$$\Lambda(S) = \{\lambda \mid \lambda \in W_2^{-1/2}(S),\ \lambda \geqslant 0\}$$

be a set of Lagrange multipliers such that

$$\sup_{\lambda\in\Lambda(S)} \langle -\lambda, \varphi\rangle = \begin{cases} 0, & \varphi \in V^*(S) \\ +\infty, & \varphi \notin V^*(S) \end{cases}$$

where $\langle,\rangle$ is the duality ratio in $W_2^{*^{1/2}}(S) \times W_2^{-1/2}(S)$.

Then the problem of determining

$$\inf_\varphi \sup_\lambda \{F_0(\varphi) + \langle -\lambda, \varphi\rangle\} \tag{2.5}$$

(the direct formulation) is equivalent (as is confirmed directly) to the initial problem (2.4).

Here and henceforth $\inf_\varphi$ and $\sup_\lambda$ mean

$$\operatorname*{Inf}_{\varphi\in W_2^{*^{1/2}}(S)}, \quad \sup_{\lambda\in\Lambda(S)}$$

Following /7/ (pp.214-216), it can be shown that the problem dual to (2.5) will be the problem of determining

$$\sup_\lambda \inf_\varphi \{F_0(\varphi) + \langle -\lambda, \varphi\rangle\} \tag{2.6}$$

and the saddle point $\{\varphi_0, \lambda_0\} \in W_2^{*^{1/2}}(S) \times \Lambda(S)$ of the Lagrangian

$$L(\varphi, \lambda) = F_0(\varphi) - \langle \lambda, \varphi\rangle \tag{2.7}$$

is determined by the condition

$$\begin{gathered} F_0(\varphi_0) - \langle \lambda, \varphi_0\rangle \leqslant F_0(\varphi_0) - \langle \lambda_0, \varphi_0\rangle \leqslant F_0(\varphi) - \langle \lambda_0, \varphi\rangle \\ \forall\varphi \in W_2^{*^{1/2}}(S), \quad \forall\lambda \in \Lambda(S) \\ \varphi_0 \geqslant 0, \quad \lambda_0 \geqslant 0 \Rightarrow \langle \varphi_0, \lambda_0\rangle = 0 \end{gathered} \tag{2.8}$$

for whose proof the Hahn-Banach theorem is used /7/.

It follows from (2.8) that the relationships

$$\min_\varphi \max_\lambda L(\varphi,\lambda) = \max_\lambda \min_\varphi L(\varphi, \lambda) = F_0(\varphi_0) \tag{2.9}$$

hold, where the quantity $F_0(\varphi_0) = -{}^1/{}_2 S\partial_\nu\varphi_0\varphi_0 ds$ is found from the generalized Euler-Lagrange equation $F_0'(\varphi_0, \psi) = 0$, $\forall\psi \in W_2^{*^{1/2}}(S)$ for the functional $F_0(\varphi)$ (2.2). Interpretation of inequalities (2.8) shows that the argument φ_0 of the saddle point $\{\varphi_0, \lambda_0\}$ is a solution of problem (2.3).

Indeed, it follows from the right inequality in (2.8) that the (Fréchet) derivative of $L(\varphi, \lambda_0)$ vanishes at the point φ_0, which yields, for sufficiently regular functions φ and λ

$$\int \partial_\nu\varphi_0\psi\, ds + \int \partial_\nu u^*\psi\, ds - \int \lambda_0\psi\, ds = 0, \quad \forall\psi \in W_2^{*^{1/2}} \tag{2.10}$$

It hence follows that $\partial_\nu\varphi_0 + \partial_\nu u^* = \lambda_0 \geqslant 0$; the inequality

$$\langle \lambda, \varphi_0\rangle \geqslant \langle \lambda_0, \varphi_0\rangle, \quad \forall\lambda \geqslant 0 \tag{2.11}$$

follows from the left inequality in (2.8).

Consequently, since $\langle \lambda_0, \varphi_0\rangle = 0$, we have $\varphi_0 \geqslant 0$ and $\varphi_0[\partial_\nu\varphi_0 + \partial_\nu u^*] = 0$.

Therefore, the conditions on the boundary in (2.3) are satisfied, and satisfcation of the equation $A\varphi_0 = 0$ in G follows from the trace belonging to $\varphi_0|_S \in W_2^{*^{1/2}}(S)$. We also note that the equality $\partial_\nu\varphi_0 + \partial_\nu u^* = \lambda_0$ resulting from (2.10) yields an interpretation of the Lagrange multiplier λ_0 which has a definite mechanical meaning in the Signorini problem for the linear elasticity theory operator (see below).

We will henceforth examine a solution of the dual problem (2.6) in the form

$$\max_\lambda \min_\varphi L(\varphi, \lambda) \tag{2.12}$$

3. The following algorithm is proposed for solving problem (2.12).

1°. For fixed $\lambda > 0$ ($\lambda \neq \partial_\nu u^*$) the problem $\min_\varphi L(\varphi, \lambda)$ is solved, which reduces to solving a variational equation of the form (2.10)

$$\int \partial_\nu\varphi_\lambda\psi\, ds + \int \partial_\nu u^*\psi\, ds - \int \lambda\psi\, ds = 0, \quad \forall\psi \in W_2^{*^{1/2}}(S) \tag{3.1}$$

The approximate Ritz solution

$$\varphi_{\lambda n} \equiv \varphi_n(\lambda, x) = \sum_{i=1}^n a_i(\lambda)\beta_i(x), \quad x \in G \tag{3.2}$$

is constructed in coordinate functions in the form of double-layer potentials

$$\beta_i(x) = -(4\pi)^{-1}\int_S \frac{\partial}{\partial\nu}\Gamma(x, y)\psi_i(y)\,ds_y, \quad i = 1, 2, \ldots$$

where $\{\psi_i\}$ is a sequence of fairly smooth linearly independent functions defined at the points $y \in S$; completeness of $\{\psi_i\}$ in $L_2(S)$ is assumed. The functions $\beta_i(x)$ are allowable functions of the problem $\min_\varphi L(\varphi, \lambda)$ by virtue of the known /8/ properties of the boundary potentials, namely

$$A\beta_i(x) = 0, \ \forall x \in G, \ \beta_i|_S = \psi_i(y), \ \forall y \in S$$

Therefore $\beta_i|_S \in W_2^{*1/2}(S)$ and the approximate solution (3.2) has the following form at points of the boundary S:

$$\varphi_n(\lambda, y) = \sum_{i=1}^n a_i(\lambda)\psi_i(y)$$

We obtain a system of linear equations to determine the coefficients a_i (for each fixed $\lambda > 0, \lambda \neq \partial_\nu u^*$) from (3.1)

$$\sum_{i=1}^n a_i \int \partial_\nu\psi_i\psi_k\,ds = -\int \partial_\nu u^*\psi_k\,ds + \int \lambda\psi_k\,ds, \quad k = 1, \ldots, n \tag{3.3}$$

The matrix of this system with the elements

$$\int \partial_\nu\psi_i\psi_k\,ds = \int \partial_\nu\beta_i\beta_k\,ds = [\beta_i, \beta_k]_{1/2, S}$$

where $[,]_{1/2, S}$ is the scalar product in $W_2^{*1/2}(S)$ corresponding to the norm (2.1), is symmetric and positive-definite. Therefore, system (3.3) is uniquely solvable. Hence, the first part of the algorithm is realized.

2°. The problem $\max_\lambda L(\varphi_\lambda, \lambda)$ is solved, where $L(\varphi_\lambda, \lambda) = \min_\varphi L(\varphi, \lambda)$ and $\varphi_\lambda \equiv \varphi(\lambda)$ is the argument of the saddle point for a fixed Lagrange multiplier $\lambda > 0$; $\varphi_\lambda \in W_2^{*1/2}(S)$.

Let us calculate $L(\varphi_\lambda, \lambda)$. For $\psi = \varphi_\lambda$ we obtain from (3.1)

$$\int \partial_\nu\varphi_\lambda\varphi_\lambda\,ds = \int \lambda\varphi_\lambda\,ds - \int \partial_\nu u^*\varphi_\lambda\,ds \tag{3.4}$$

We then have from (2.7)

$$L(\varphi_\lambda, \lambda) = {}^1/_2\int \partial_\nu\varphi_\lambda\varphi_\lambda\,ds + \int \partial_\nu u^*\varphi_\lambda\,ds - \int \lambda\varphi_\lambda\,ds = -{}^1/_2\int \partial_\nu\varphi_\lambda\varphi_\lambda\,ds = \min_\varphi L(\varphi, \lambda)$$

Therefore, taking account of Eq.(3.4) the dual problem (2.12) reduces to minimization problem

$$\max_\lambda L(\varphi_\lambda, \lambda) = \max_\lambda\left\{-{}^1/_2\left(\langle\lambda, \varphi_\lambda\rangle - \int \partial_\nu u^*\varphi_\lambda\,ds\right)\right\} = -{}^1/_2\min_\lambda\left(\langle\varphi_\lambda, \lambda\rangle - \int \partial_\nu u^*\varphi_\lambda\,ds\right) \tag{3.5}$$

where φ_λ are determined from the variational Eq.(3.1) for the set $\{\lambda\}$, $\lambda > 0$, $\lambda \neq \partial_\nu u^*$.

We will use the notation

$$\Phi(\lambda) = \langle\lambda, \varphi_\lambda\rangle - \int \partial_\nu u^*\varphi_\lambda\partial s \quad (\Phi(\lambda) \not\equiv 0)$$

By virtue of (3.4) (for sufficient regularity of λ) we have

$$\Phi(\lambda) = \int \partial_\nu\varphi_\lambda\varphi_\lambda\,ds = \|\varphi_\lambda\|^2_{1/2, S} > 0, \quad \forall\varphi_\lambda \neq 0$$

(see (2.1)), i.e., $\Phi(\lambda)$ is a strictly convex functional and $\Lambda(S)$ is a closed convex set /7/ in dual space $W_2^{-1/2}(S)$. Therefore, the problem $\min_\lambda \Phi(\lambda)$ is uniquely solvable.

It can be established that $\min_\lambda \Phi(\lambda) = \Phi(\lambda_0)$.

Indeed, it follows from the left inequality in (2.8) for the function $L(\varphi_\lambda, \lambda)$ with $\varphi_\lambda = \varphi_0$ that $\max_\lambda L(\varphi_0, \lambda)$, which means (by virtue of (3.5)) also $\min_\lambda \Phi(\lambda)$ is achieved at the point $\lambda_0 \in \Lambda(S)$ and equals

$$\Phi(\lambda_0) = \int \partial_\nu \varphi_0 \varphi_0 \, ds$$

Finally, taking account of the factor $-{}^1/_2$ in (3.5), we obtain

$$\max_\lambda \min_\varphi L(\varphi, \lambda) = -{}^1/_2 \int \partial_\nu \varphi_0 \varphi_0 \, ds = F_0(\varphi_0)$$

which corresponds to (2.9).

To solve the problem $\max_\lambda L(\varphi_\lambda, \lambda)$, which reduces to the problem $\min_\lambda \Phi(\lambda)$, an algorithm of the descent method is used, which is a generalization of the Frank-Wolf method for the case of convex constraints $\lambda > 0$ (/7/, p.130). The passage from the iteration λ_m to λ_{m+1} is realized as follows: on selecting the initial approximation $\lambda^{(0)} > 0$

$$\lambda_{m+1} = \lambda_m + \rho_m \mu_m, \quad \lambda_m > 0, \quad m = 1, 2, \ldots \tag{3.6}$$

For $\lambda = \lambda_m > 0$ let the Ritz approximations $\{\varphi_{\lambda_m n}\}_{n=1,2,\ldots}$ be constructed according to Sect. 1°. In order for the pair $\{\varphi_{\lambda_m n}, \lambda_m\}$ to be an approximate saddle point $L(\varphi, \lambda)$, satisfaction of the following inequality is necessary (analogous to (2.11))

$$\langle \lambda_m, \varphi_{\lambda_m n} \rangle \leqslant \langle \lambda, \varphi_{\lambda_m n} \rangle, \quad \forall \lambda \in \Lambda(S) \tag{3.7}$$

which ensures satisfaction of the left-hand side of relationships (2.8) defining the saddle point. The right-hand side of this relationship is satisfied since $\varphi_{\lambda_m n}$ is a Ritz approximate solution of the problem of finding $\min_\varphi L(\varphi, \lambda_m)$. The descent direction μ_m and the step ρ_m in the iteration process (3.6) are selected according to well-known recommendations (/7/, p.227). First $\beta_m \in \Lambda(S)$, is determined such that inequality (3.7) is satisfied in the form

$$\langle \varphi_{\lambda_m n}, \beta_m - \beta \rangle \leqslant 0, \quad \forall \beta \in \Lambda(S)$$

in particular for $\beta = \lambda_m$. The step

$$\rho_m = \min\{1, -c_0^{-1} \langle \varphi_{\lambda_m n}, \beta_m - \lambda_m \rangle\}, \quad c_0 > 0 \quad (\rho_m \neq 1)$$

is then calculated, where c_0 is a fairly large fixed number. Therefore, we obtain the iteration λ_{m+1} according to (3.6) for the descent direction $\mu_m = \beta_m - \lambda_m$ and the step ρ_m. The condition to halt the iteration process (3.6), which is of practical interest for solving the Signorini problem of elasticity theory in Sect.1, is presented in Sect.5.

4. We will now justify the algorithm proposed. It follows from the constructions presented in Sect.3 that for each fixed $\lambda \in \{\lambda\}$ approximations $\varphi_{\lambda n}$ of the form (3.2) are approximations of the Ritz process of the problem $\min_\varphi L(\varphi, \lambda)$. Indeed, it is sufficient to confirm satisfaction of the conditions to which the basis functions are subject in the Ritz process (/5/, p.96). It should here to taken into account that linearity of the integral operator of the boundary double-layer potential type and the assumed completeness of the sequence $\{\psi_i\}$ in $L_2(S)$ ensure the basic character in $L_2(S)$ for the sequence $\{\beta_i\}$. Thus:

a) For any n, the elements $\beta_1, \beta_2, \ldots, \beta_n$ are linearly independent;

b) The traces $\beta_i|_S$ are elements of the energy space of the functions $W_2^{*1/2}(S)$; indeed (see Sect.2), the subspace $W_2^{*1/2}(S)$ allotted to the norm (2.1) can be considered as an energy space of traces in S for sufficiently smooth functions satisfying the equation $A\varphi = 0$ in G;

c) The sequence $\{\beta_i|_S\}$ is complete in the norm in $W_2^{*1/2}(S)$; this follows from the completeness of $\{\beta_i|_S\}$ in $L_2(S)$ and positive-definiteness of the form $\int \partial_\nu \varphi \varphi \partial s$ on the basis of known results (/5/,p.366, Theorem 1).

According to Sect.2, the $\min_\varphi L(\varphi, \lambda)$ (for fixed λ) is achieved at the point $\varphi_\lambda \in W_2^{*1/2}(S)$. Then by using the results on the convergence of the Ritz process from /5/ (on the basis of a)-c)), it can be asserted that the convergence

$$\lim_{n\to\infty} \| \varphi_{\lambda n} - \varphi_\lambda \|_{1/2,\, S} = 0 \tag{4.1}$$

holds.

According to /7/, the iteration process (3.6) of the solution of the problem of finding $\max_\lambda L(\varphi_\lambda, \lambda)$ converges so that

$$\lim_{m\to\infty} \| \lambda_m - \lambda_0 \|_{-1/2,\, S} = 0, \quad \|\cdot\|_{-1/2,\, S} = \|\cdot\|_{W_2^{-1/2}(S)} \tag{4.2}$$

where λ_0 is the argument of the saddle point $\{\varphi_0, \lambda_0\}$.

A theorem on the convergence of the algorithm is proved on the basis of (4.1) and (4.2).

Theorem. A family of problem $\min_\varphi L(\varphi, \lambda_m)$ and a set of approximate solutions $\{\varphi\}_{\lambda_m n} \equiv \{\varphi_n(\lambda_m, x)\}_{n,\, m=1,2,\ldots}$, such that for each fixed $\lambda = \lambda_m \in \{\lambda_m\}$ the convergence $\varphi_{\lambda_m n} \to \varphi_{\lambda_m}$ occurs as $n \to \infty$ in the sense of (4.1), correspond to the sequence of iterations $\{\lambda_m\}_{m=1,2,\ldots}$. Then if $\lambda_m \to \lambda_0$ as $m \to \infty$ in the sense of (4.2), then $\varphi_{\lambda_m} \to \varphi_0$ as $m \to \infty$ in the norm in $W_2^{*1/2}(S)$.

Proof. We obtain two equalities, respectively, from (2.10) for $\psi = \varphi - \varphi_0$ and from (3.1) for $\psi = \varphi - \varphi_{\lambda_m}$. We set $\varphi = \varphi_{\lambda_m}$ in the first equality, and $\varphi = \varphi_0$ in the second and we subtract. We then obtain

$$\int (\partial_\nu \varphi_0 - \partial_\nu \varphi_{\lambda_m})(\varphi_0 - \varphi_{\lambda_m})\, ds = \int (\lambda_0 - \lambda_m)(\varphi_0 - \varphi_{\lambda_m})\, ds \tag{4.3}$$

By virtue of (2.1) the left-hand side of (4.3) equals $\| \varphi_0 - \varphi_{\lambda_m} \|^2_{1/2, S}$.

Using the generalized Schwartz inequality for the right-hand side of (4.3), we obtain the inequality

$$\| \varphi_0 - \varphi_{\lambda_m} \|_{1/2, S} \leqslant \| \lambda_0 - \lambda_m \|_{-1/2, S}$$

from which on satisfying (4.2) it follows that $\lim \| \varphi_0 - \varphi_{\lambda_m} \|_{1/2, S} = 0$ as $m \to \infty$.

The convergence $\varphi_{\lambda_m} \to \varphi_0$ as $m \to \infty$ also occurs in the norm in the Sobolev class of functions $W_2^1(G)$ to which the generalized solutions of boundary value problems for second-order equations belong.

Indeed, for functions satisfying the equation $A\varphi = 0$ in G the equation

$$B(\varphi) = \int \partial_\nu \varphi \varphi\, ds = \| \varphi \|^2_{1/2, S}$$

follows from Green's formula and (2.1).

The estimate /5/

$$B(\varphi) \geqslant c \| \varphi \|^2_{1, G}, \quad c > 0, \ \| \cdot \|^2_{1, G} = \| \cdot \|_{W_2^1(G)}$$

holds for the positive-definite quadratic form $B(\varphi)$, therefore from $\| \varphi \|^2_{1/2, S} \geqslant c \| \varphi \|^2_{1, G}$ and for $\| \varphi_0 - \varphi_{\lambda_m} \|_{1/2, S} \to 0 \ (m \to \infty)$ the convergence $\| \varphi_0 - \varphi_{\lambda_m} \|_{1, G} \to 0$ follows as $m \to \infty$.

The algorithm constructed can be considered as a modification of the well-known algorithm (see /3, 4/, say) for finding the saddle points of Lagrangians by Udzawa-Arrow-Hurwitz since for alternate utilization of the Ritz approximation $\{\varphi_n\}_{n=1,2,\ldots}$ and the iterations $\{\lambda_m\}_{m=1,2,\ldots}$ the value of the functional $L(\varphi_{\lambda_m n}, \lambda_m)$ tends to $L(\varphi_0, \lambda_0)$ as $m, n \to \infty$, where $\{\varphi_0, \lambda_0\}$ is the saddle point of the Lagrangian $L(\varphi, \lambda)$.

We also note that by virtue of the equation $\lambda = \partial_\nu \varphi_{\lambda n} + \partial_\nu u^*$ which follows from (3.1) where $\varphi_{\lambda_n}|_S = \Sigma a_i(\lambda) \psi_i(y)$, the expansion of the multipliers λ in a system of functions $\{\partial_\nu \psi_i\}$ is of definite interest for an appropriate foundation. A similar algorithm for constructing approximate values of the saddle points is proposed and proved in /9/.

In general the possibilities of applying the proposed algorithm are constrained to boundary value problems for which Green's function exists, but if it is taken into account that Green's function is required in explicit form to obtain a solution at points of the domain in the boundary values found, then contact problems of linear elasticity theory in which the displacement and stress distribution must be found in the contact zone are a possible domain of application of the algorithm. For such problems satisfaction of the conditions imposed on the data of the problems for which Green's function exists /8/ is sufficient, and its construction in explicit form is not essential.

In connection with this remark we note that the duality different from the algorithm elucidated, which utilizes Green's function for the integral relation of the contact pressure with the displacements in the contact zone, is represented in /10/ for the solution of contact problems of elasticity theory.

5. The constructions elucidated in Sects.2-4 are extended to the generalized Signorini problem for the linear elasticity theory operator (in the formulation elucidated in Sect.1, see /2/ also), which reduces to the problem of minimizing the boundary functional (1.1) in the set $\mathbf{V}^*(S_1)$. The solution $\boldsymbol{\varphi}_0 \in \mathbf{V}^*(S_1)$ of this problem satisfies the variational inequality /2/

$$\int_{S_1} \mathbf{t}^{(\nu)}(\boldsymbol{\varphi}_0)(\mathbf{v} - \boldsymbol{\varphi}_0)\, ds \geqslant -\int_{S_1} \mathbf{t}^{(\nu)}(\mathbf{u}^*)(\mathbf{v} - \boldsymbol{\varphi}_0)\, ds, \quad \forall \mathbf{v} \in \mathbf{V}^*(S_1)$$

Since the set $\mathbf{V}^*(S_1)$ (see (1.2)) is a closed convex cone /2/ with apex at the origin, this inequality is equivalent /3/ to the relationships

$$\int \mathbf{t}^{(\nu)}(\boldsymbol{\varphi}_0)\, \mathbf{v}\, ds \geqslant -\int \mathbf{t}^{(\nu)}(\mathbf{u}^*)\, \mathbf{v}\, ds, \quad \forall \mathbf{v} \in \mathbf{V}^*(S_1)$$

$$\int \mathbf{t}^{(\nu)}(\boldsymbol{\varphi}_0)\, \boldsymbol{\varphi}_0\, ds = -\int \mathbf{t}^{(\nu)}(\mathbf{u}^*)\, \boldsymbol{\varphi}_0\, ds, \quad \boldsymbol{\varphi}_0 \in \mathbf{V}^*(S_1)$$

(here the integration is over S_1). By virtue of (1.4) $\int t^{(\nu)}(u^*)\varphi_0^{(\nu)}ds \geqslant 0$ follows from the second relationship. Since $\varphi_0^{(\nu)}|_{S_1} \geqslant 0$, this inequality id satisfied if the given stress vector satisfies the condition $t^{(\nu)}(u^*)|_{S_1} < 0$ in the contact zone, which corresponds to the solvability condition for the variational problem for the functional (1.1) in the set $V^*(S_1)$ /1/:

$$\int_{S_1} t^{(\nu)}(u^*)\rho\, ds \leqslant 0, \quad \forall \rho \in \mathbf{R} \cap \mathbf{V}^*(S_1)$$

where $\mathbf{R}$, the subspace of stiff displacements, is the kernel of the quadratic form of linear elasticity theory $2\int_G W(\varphi)\,dG$, and the equality sign in this condition will hold only if ρ is vector of the bilateral displacements of points of the contact boundary S_1.

The mechanical interpretation of the Lagrange multiplier λ_0 (see Sect.2) follows from the equality $t^{(\nu)}(\varphi_0) + t^{(\nu)}(u^*) = \lambda_0$. Namely, since $t^{(\nu)}(u^*) < 0$ and $\lambda_0 \geqslant 0$ (see (2.8)), λ_0 is the intensity of the distributed normal reference reaction at points of the set (unknown a priori) $S_1^\circ \subset S_1$ in which body contact with the reference surface exists and $\varphi_0^{(\nu)}|_{S_1^\circ} = 0$.

We will investigate the possibilities of the algorithm constructed in Sects.3, 4 from the viewpoint of determining the stress in the zone of possible contact S_1. For a certain $\lambda = \lambda_m > 0$ let the Ritz approximations $\{\varphi_{\lambda_m n}\}_{n=1,2,\ldots}$ of the solution of the problem $\min_\varphi L(\varphi, \lambda_m)$ be constructed from a variational equation of the form (3.1) in which the derivatives $\partial_\nu\varphi_\lambda$ and $\partial_\nu u^*$ are, respectively, the vectors of the surface stresses $t^{(\nu)}(\varphi_\lambda)$ and $t^{(\nu)}(u^*)$. We show that the sequence $\{t^{(\nu)}(\varphi_{\lambda_m n})\}_{m=1,2,\ldots;\ n=1,2,\ldots}$ converges as $n, m \to \infty$ in the sense $W_2^{-1/2}(S_1)$ to $t^{(\nu)}(\varphi_0)$—the stress vector in the contact zone S_1 that corresponds to the exact solution $\varphi_0 \in V^*(S_1)$ of the unilateral variational problem for the functional (1.1). Indeed, according to (4.1), $|\varphi_{\lambda_m n} - \varphi_{\lambda_m}| \to 0$ holds for each λ_m as $n \to \infty$, where the norm is defined according to (1.4). Then by virtue of the imbedding theorem $W_2^{*1/2}(S) \subset W_2^{-1/2}(S)$ and the estimate /2, 6/ $\| t^{(\nu)}(\varphi)\|_{-1/2, S_1} \leqslant c_1 |\varphi|_{1/2, S_1}$, $c_1 > 0$ we have $\| t^{(\nu)}(\varphi_{\lambda_m n}) - t^{(\nu)}(\varphi_{\lambda_m})\|_{-1/2, S_1} \to 0$ as $n \to \infty$. Furthermore, since the equality $\lambda_0 - \lambda_m = t^{(\nu)}(\varphi_0) - t^{(\nu)}(\varphi_{\lambda_m})$, holds, then from (see (4.2)) $\|\lambda_0 - \lambda_m\|_{-1/2, S_1} \to 0$ as $m \to \infty$, the convergence of

$$\| t^{(\nu)}(\varphi_0) - t^{(\nu)}(\varphi_{\lambda_m})\|_{-1/2, S_1} \longrightarrow 0.$$

as follows.

Remark. Certain complications of a technical nature are caused by the condition $t^{(\nu)}(\varphi)|_{S_2} = 0$ (see (1.3)) to which the allowable functions of the variational problem for the functional (1.1) should be subjected. But if this functional is taken in the form (first integral over all S)

$$F_1(\varphi) = \frac{1}{2}\int_S t^{(\nu)}(\varphi)\varphi\, ds + \int_{S_1} t^{(\nu)}(u^*)\varphi\, ds$$

the condition mentioned is a natural condition for the minimization of $F_1(\varphi)$.

Realization of the proposed algorithm was examined in an example of the classical plane contact problem of the insertion of an absolutely stiff stamp into an elastic half-plane (without taking account of friction). The normal stress vector in the contact zone $t^{(\nu)}(u)$, which is considered known and should satisfy the condition $t^{(\nu)}(u)|_{S_1} < 0$ in the formulation of the unilateral boundary value problem (1.5) (the vector $t^{(\nu)}(u)$ is not associated here with the formulation of the auxiliary mixed problem of elasticity theory for the displacement vector u^*, see Sect.1), was here given thus: $t^{(\nu)}(u) = -\mathbf{p}$, where $\mathbf{p}(y) > 0$ is a function of the normal contact pressure under the stamp with definite surface geometry of the stamp in the contact domain during the action of a force $P = \int \mathbf{p}(y)\,dy$, on the stamp, where the integration is over the width $2a$ of the possible contact zone that is symmetric relative to the stamp axis /11/. For certain surface shapes constraining the stamp base, the functions $\mathbf{p}(y)$ obtained by methods of the theory of complex variable functions are presented in /11/. For a given vector $t^{(\nu)}(u) = -\mathbf{p}$ in the formulation of the above-mentioned contact problem, numerical analysis of the algorithm elucidated in Sect.3 for the solution of this problem reduces to an analysis of the approach of the integral

$$\langle \varphi_{\lambda_m n}, \lambda_m\rangle = \int_{S_1} \varphi_{\lambda_m n}(t^{(\nu)}(\varphi_{\lambda_m n}) - \mathbf{p})\, ds \qquad (5.1)$$

to zero as the number of iterations m and the number of Ritz approximations n increase.

Theoretically the convergence $\langle\varphi_{\lambda_m n}, \lambda_m\rangle \to \langle\varphi_0, \lambda_0\rangle = 0$ holds as $m, n \to \infty$, since from the convergence $\varphi_{\lambda_m n} \to \varphi_{\lambda_m}$, $\forall\lambda_m \in \Lambda(S_1)$ in the sense of (4.1) and the convergence $\lambda_m \to \lambda_0$ in the sense of (4.2) as well as the convergence $\varphi_{\lambda_m} \underset{m\to\infty}{\to} \varphi_0$ in the sense of $W_2^{*1/2}(S_1)$ (see the Theorem of Sect.4), there follows the above-mentioned convergence in the sense of the ratio of the daulity in $W_2^{*1/2}(S_1) \times W_2^{-1/2}(S_1)$.

In constructing the Ritz approximation, the contact boundary S_1 was approximated by isoparametric curvilinear second-order boundary elenents (BE). The construction and foundation of the boundary-element approximation of the variational problem for a boundary functional of the form (1.1) by using basis functions of the double-layer boundary potential type is presented in /12/.

The condition to terminate the iteration process was given thus

$$\sum_{i=1}^{n} \left| \int_{\Delta s_i} \varphi_{\lambda_m n}^{(i)} [t^{(\nu_i)}(\varphi_{\lambda_m n}^{(i)}) - p]\, ds_i \right| < \varepsilon \tag{5.2}$$

where ε is a given positive number governing the required accuracy of the iteration process in λ_m for a fixed number n of BE Δs_i.

For a circular stamp contained within the limits of the possible contact zone S_1 of the curve $f(y) = y^2/(2R)$ (under the assumption that the radius of the stamp base is large compared with the size of the contact area), the function of the given contact pressure $p(y)$ was taken from /11, p.65/. Two modifications of the contact boundary partition into BE were examined for the greatest assumed depth of stamp insertion $h = 0.02R$ (along the stamp axis of symmetry) and a corresponding possible contact zone halfwidth $a = 0.2R$. For six elements and $\varepsilon = 5 \times 10^{-2}$ in condition (5.2), the greatest error (at the point $y = 0$ on the stamp axis of symmetry) in the values of p and $t^{(\nu)}(\varphi_{\lambda_m n})$ was $\delta = 16\%$ $(m = 14)$. The following values of the error were obtained for twelve elements: $\delta \approx 14.5\%$ for $\varepsilon = 5 \times 10^{-2}$ $(m = 18)$; $\delta \approx 8\%$ for $\varepsilon = 10^{-2}$ $(m = 29)$; $\delta \approx 1.5\%$ for $\varepsilon = 10^{-3}$ $(m = 55)$; the calculations were performed on an ES-1022 computer. It is established that an increase in the number of iterations m affects the decrease of δ to a greater degree than an increase in the number n of BE.

The example considered is substantially confirmatory for the proposed algorithm in the sense that the solution $p(y)$ of the contact problem by the method of the theory of complex variable functions /11/ is compared with the solution $t^{(\nu)}(\varphi_{\lambda n})$ of this problem as a unilateral variational problem for the boundary functional (1.1).

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